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Note on the notion of incompressibility in theories of porous and granular materials

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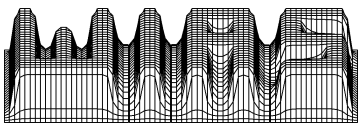

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Abstract

We present a simple two-component model of a porous material based on the constraint assumption that the so-called true components are incompressible. In my previous work on this subject [1] I pointed out that many such models are not thermodynamically admissible. Namely the second law of thermodynamics led to the conclusion that an additional field of reaction force on the constraint cannot be introduced, and, consequently, the set of field equations was overdetermined. However I speculated as well that an extension of the set of variables may lead to thermodynamic admissibility. Indeed an example presented in this paper supports this speculation. According to results of this work it seems to be neccessary to introduce higher gradients to multicomponent models with constraints in order to satisfy the second law of thermodynamics.

1 Introduction

In the paper [1] I pointed out that constraint conditions in multicomponent systems may not lead to thermodynamically admissible models. I indicated that the thermodynamic admissibility depends on a choice of constitutive variables, and I speculated that some theories with higher gradients may lead to models in which constraints such as the incompressibility of true components may be thermodynamically admissible.

The above question has a practical bearing as numerous models of porous, and granular materials do indeed rely on such assumptions. In this note I present a simple example of such a gradient theory of a two-component system which is thermodynamically admissible in spite of incompressibility constraints for true components. In order to appreciate better the argument the constitutive relations are chosen in a particularly simple form.

We rely on a lagrangian description of motion of both components (e.g.: [2,3]). We assume processes to be *isothermal*. In such a case the fields of the model are as follows

n - void fraction of the solid component (porosity),

v_k^F - velocity of the fluid component,

v_k^S - velocity of the solid component,

$F_{k\alpha}^S$ - deformation gradient of the solid component.

We use the cartesian reference systems for both lagrangian coordinates X^α (α=1,2,3), and eulerian coordinates x^k ($k = 1, 2, 3$). All fields are assumed to be defined on a reference configuration of the solid component for which $F_{k\alpha}^S = \delta_{k\alpha}$.

2 Balance equations

We construct field equations on the basis of partial balance equations.

Partial mass densities ρ_t^S, ρ_t^F in a current configuration, partial mass densities ρ^S, ρ^F in the reference configuration, and the so-called true mass densities ρ^{SR}, ρ^{FR} are connected by the relations

$$\begin{aligned} \rho_t^S &= \rho^S J^{S-1} = (1-n)\,\rho^{SR}, \qquad \rho^{SR} = const., \\ \rho_t^F &= \rho^F J^{S-1} = n\rho^{FR}, \qquad \rho^{FR} = const., \qquad J^S := \det F_{k\alpha}^S, \end{aligned} \tag{2.1}$$

where the index t denotes the current configuration. The assumption that the true mass densities are constant introduces a constraint into the model.

The partial mass balance equations in the lagrangian description have the following form

$$\frac{\partial \rho^S}{\partial t} = 0, \qquad \frac{\partial \rho^F}{\partial t} + \frac{\partial \left(\rho^F X_\alpha^{\prime F}\right)}{\partial X^\alpha} = 0, \qquad X_\alpha^{\prime F} := F_{\alpha k}^{S-1}\left(v_k^F - v_k^S\right), \tag{2.2}$$

where $X_\alpha^{\prime F}$ denotes the lagrangian velocity of the fluid with respect to the solid in the reference configuration. Bearing the constraint of constant true mass densities specified in (2.1) in mind we can reformulate these equations in the following manner

$$\begin{aligned} -\frac{\partial n}{\partial t} + (1-n)\,F_{\alpha k}^{S-1}\frac{\partial v_k^S}{\partial X^\alpha} &= 0, \\ F_{\alpha k}^{S-1}\frac{\partial}{\partial X^\alpha}\left[(1-n)\,v_k^S + n v_k^F\right] &= 0. \end{aligned} \tag{2.3}$$

The momentum balance equations have the following form

$$\begin{aligned} \rho^S \frac{\partial v_k^S}{\partial t} &= \frac{\partial P_{k\alpha}^S}{\partial X^\alpha} - p_k^*, \\ \rho^F\left(\frac{\partial v_k^F}{\partial t} + X_\alpha^{\prime F}\frac{\partial v_k^F}{\partial X^\alpha}\right) &= \frac{\partial P_{k\alpha}^F}{\partial X^\alpha} + p_k^*. \end{aligned} \tag{2.4}$$

In these equations $P_{k\alpha}^S$, and $P_{k\alpha}^F$ denote partial Piola-Kirchhoff stress tensors in the solid and fluid component, respectively, and p_k^* is the source of momentum.

The partial internal energy balance equations have, in general, the following form

$$\begin{aligned} \rho^S \frac{\partial \varepsilon^S}{\partial t} + \frac{\partial Q^S_\alpha}{\partial X^\alpha} &= P^S_{k\alpha} \frac{\partial v^S_k}{\partial X^\alpha} + p^*_k v^S_k, \\ \rho^F \left(\frac{\partial \varepsilon^F}{\partial t} + X'^F_\alpha \frac{\partial \varepsilon^F}{\partial X^\alpha} \right) + \frac{\partial Q^F_\alpha}{\partial X^\alpha} &= P^F_{k\alpha} \frac{\partial v^F_k}{\partial X^\alpha} - p^*_k v^F_k, \end{aligned} \tag{2.5}$$

where Q^S_α, and Q^F_α denote the partial heat flux vectors in the solid and fluid component, respectively.

Finally, the balance equations for partial entropies are

$$\begin{aligned} \rho^S \frac{\partial \eta^S}{\partial t} + \frac{\partial H^S_\alpha}{\partial X^\alpha} &= \eta^{*S}, \\ \rho^F \left(\frac{\partial \eta^F}{\partial t} + X'^F_\alpha \frac{\partial \eta^F}{\partial X^\alpha} \right) + \frac{\partial H^F_\alpha}{\partial X^\alpha} &= \eta^{*F}, \end{aligned} \tag{2.6}$$

where H^S_α , and H^F_α denote the partial entropy fluxes in the solid and fluid component, and η^{*S}, η^{*F} are the partial entropy sources, respectively.

For the purpose of this work we make a simplifying assumption that partial heat fluxes, and partial entropy fluxes are connected by the classical relations

$$H^S_\alpha = \frac{Q^S_\alpha}{T}, \qquad H^F_\alpha = \frac{Q^F_\alpha}{T}, \tag{2.7}$$

where T denotes the absolute temperature.

For the future use we combine the entropy balance equations, and the energy balance equations in the form of the following relation

$$\begin{aligned} & \rho^S \frac{\partial \psi^S}{\partial t} + \rho^F \left(\frac{\partial \psi^F}{\partial t} + X'^F_\alpha \frac{\partial \psi^F}{\partial X^\alpha} \right) - \\ & - P^S_{k\alpha} \frac{\partial v^S_k}{\partial X^\alpha} - P^F_{k\alpha} \frac{\partial v^F_k}{\partial X^\alpha} + p^*_k \left(v^F_k - v^S_k \right) + \\ & + \rho^S \eta^S \frac{\partial T}{\partial t} + \rho^F \eta^F \left(\frac{\partial T}{\partial t} + X'^F_\alpha \frac{\partial T}{\partial X^\alpha} \right) - \frac{1}{T} \left(Q^S_\alpha + Q^F_\alpha \right) \frac{\partial T}{\partial X^\alpha} \\ = \; & - T \left(\eta^{*S} + \eta^{*F} \right), \end{aligned} \tag{2.8}$$

where we have introduced the partial Helmholtz free energies

$$\psi^S := \varepsilon^S - T\eta^S, \qquad \psi^F := \varepsilon^F - T\eta^F. \tag{2.9}$$

3 Second law of thermodynamics

In order to formulate field equations for the fields listed in the introduction we have to close the problem by constitutive relations for the constitutive quantities appearing in the equations (2.4), i.e. for Piola-Kirchhoff stress tensors, and for the source of momentum. Simultaneously the evaluation of the second law of thermodynamics requires constitutive relations for the partial Helmholtz free energies ψ^S and ψ^F appearing in (2.8). We assume that these constitutive quantities are functions of the following constitutive variables

$$\left\{ n, \frac{\partial n}{\partial X^\alpha}, F^S_{k\alpha}, L^F_{k\alpha}, X'^F_\alpha \right\}, \qquad L^F_{k\alpha} := \frac{\partial v^F_k}{\partial X^\alpha}, \tag{3.1}$$

i.e. the fluid component is assumed to introduce the viscosity into the model. Essential properties of the model connected with the presence of constraints are not influenced by the viscosity but it is introduced because many models of practical bearing requires its presence.

For sake of simplicity we assume also that constitutive relations are linear with respect to the vector variables $\frac{\partial n}{\partial X^\alpha}, X'^F_\alpha$. It can be shown that essential results which we present further hold also under weaker assumptions (e.g. nonlinear dependence on the relative velocity X'^F_α).

Then for *isotropic* systems which we consider in this work stress tensors, and free energies are independent of these vector variables, and the source of momentum has the following structure

$$\begin{aligned} p^*_k &= -F^S_{k\alpha}\left(\pi_0\delta_{\alpha\beta} + \pi_1 C^S_{\alpha\beta} + \pi_2 C^S_{\alpha\mu}C^S_{\mu\beta}\right) X'^F_\beta - \\ &\quad -F^{S-1}_{\alpha k}\left(\nu_0\delta_{\alpha\beta} + \nu_1 C^S_{\alpha\beta} + \nu_2 C^S_{\alpha\mu}C^S_{\mu\beta}\right)\frac{\partial n}{\partial X^\beta}, \end{aligned} \tag{3.2}$$

where $C^S_{\alpha\beta} \equiv F^S_{k\alpha}F^S_{k\beta}$ denotes the right Cauchy-Green deformation tensor of the solid component. The minus signs are introduced for convenience. The coefficients π_0 to ν_2 may sitll depend on the nonvectorial variables appearing in (3.1).

We simplify the relation (3.2) even further. Namely we consider solely moderate deformations of the solid component which allows to leave out the nonlinear contributions of the deformation gradient to (3.2) (i.e. $C_{\alpha\beta} \cong \delta_{\alpha\beta}$ in the brackets of this relation). Then

$$p^*_k = -\pi\left(v^F_k - v^S_k\right) - \nu F^{S-1}_{\alpha k}\frac{\partial n}{\partial X^\alpha}. \tag{3.3}$$

where $\pi \equiv \pi_0 + \pi_1 + \pi_2$, and $\nu \equiv \nu_0 + \nu_1 + \nu_2$.

Under these constitutive assumptions the set of balance equations (2.3) and (2.4) becomes the set of field equations for the fields n, v_k^S, v_k^F. The field of deformation gradient of the solid component must satisfy the following kinematical integrability conditions

$$\frac{\partial F_{k\alpha}^S}{\partial t} = \frac{\partial v_k^S}{\partial X^\alpha}, \qquad \frac{\partial F_{k\alpha}^S}{\partial X^\beta} = \frac{\partial F_{k\beta}^S}{\partial X^\alpha}, \tag{3.4}$$

which play the role of field equations for $F_{k\alpha}^S$.

It is easy to notice that the set of field equations is *overdetermined*. This is due to the presence of two equations (2.3) for the singe field of porosity. This problem appears in all models with constraints, and it is solved by introducing an additional field of the *reaction force* on constraints. Such a force must be orthogonal to the hypersurface in the space of solutions, defined by the constraint condition if the condition describes a so-called holonomic constraint. For such constraints the reaction force does no work in real processes. The geometric constraint considered in this work is such a constraint. We introduce this reaction force by means of the second law of thermodynamics.

Solutions of field equations are assumed to satisfy identically the entropy inequality which has in the present case the following form

$$\eta^{*S} + \eta^{*F} \geq 0. \tag{3.5}$$

We use Lagrange multipliers to eliminate the limitation of this inequality to solutions of field equations. Bearing the relation (2.8) for isothermal processes in mind we obtain the following inequality

$$\begin{aligned}
&\rho^S \frac{\partial \psi^S}{\partial t} + \rho^F \left(\frac{\partial \psi^F}{\partial t} + X_\alpha^{\prime F} \frac{\partial \psi^F}{\partial X^\alpha} \right) - \\
&- P_{k\alpha}^S \frac{\partial v_k^S}{\partial X^\alpha} - P_{k\alpha}^F \frac{\partial v_k^F}{\partial X^\alpha} + p_k^* \left(v_k^F - v_k^S \right) - \\
&- \Lambda \left(-\frac{\partial n}{\partial t} + (1-n) F_{\alpha k}^{S-1} \frac{\partial v_k^S}{\partial X^\alpha} \right) - \\
&- \lambda F_{\alpha k}^{S-1} \frac{\partial}{\partial X^\alpha} \left[(1-n) v_k^S + n v_k^F \right] - \\
&- \Lambda_k^S \left(\rho^S \frac{\partial v_k^S}{\partial t} - \frac{\partial P_{k\alpha}^S}{\partial X^\alpha} + p_k^* \right) - \\
&- \Lambda_k^F \left(\rho^F \left(\frac{\partial v_k^F}{\partial t} + X_\alpha^{\prime F} \frac{\partial v_k^F}{\partial X^\alpha} \right) - \frac{\partial P_{k\alpha}^F}{\partial X^\alpha} - p_k^* \right) - \\
&- \Lambda_{k\alpha} \left(\frac{\partial F_{k\alpha}^S}{\partial t} - \frac{\partial v_k^S}{\partial X^\alpha} \right) \\
\leq \; & 0.
\end{aligned} \tag{3.6}$$

This inequality must hold for arbitrary fields with the deformation gradient satisfying the symmetry condition $(3.4)_2$.

The exploitation of this inequality is standard. It is based on the Liu's Theorem on the existence of multipliers (e.g. see: [3]). Making use of the chain rule of differentiation we obtain the inequality linear with repect to derivatives listed on the left-hand side of the relations (3.7). Consequently their coefficients must vanish, and it follows

$$
\begin{aligned}
\frac{\partial n}{\partial t} \quad &: \quad \Lambda = -\left(\rho^S \frac{\partial \psi^S}{\partial n} + \rho^F \frac{\partial \psi^F}{\partial n}\right), \\
\frac{\partial v_k^S}{\partial t}, \frac{\partial v_k^F}{\partial t} \quad &: \quad \Lambda_k^S = \Lambda_k^F = 0, \\
\frac{\partial F_{k\alpha}^S}{\partial t} \quad &: \quad \Lambda_{k\alpha} = \rho^S \frac{\partial \psi^S}{\partial F_{k\alpha}^S}, \\
\frac{\partial L_{k\alpha}^F}{\partial t} \quad &: \quad \rho^S \frac{\partial \psi^S}{\partial L_{k\alpha}^F} + \rho^F \frac{\partial \psi^F}{\partial L_{k\alpha}^F} = 0, \\
\frac{\partial n}{\partial X^\alpha} \quad &: \quad \lambda + \nu = \rho^F \frac{\partial \psi^F}{\partial n}, \\
\frac{\partial v_k^S}{\partial X^\alpha} \quad &: \quad P_{k\alpha}^S = -(1-n)\,\lambda F_{\alpha k}^{S-1} - (1-n)\,\Lambda F_{\alpha k}^{S-1} + \Lambda_{k\alpha}, \\
\frac{\partial F_{k\alpha}^S}{\partial X^\beta} \quad &: \quad \underset{\alpha,\beta}{sym}\left\{\rho^F X_\alpha'^F \frac{\partial \psi^F}{\partial F_{k\beta}^S}\right\} = 0, \\
\frac{\partial L_{k\alpha}^F}{\partial X^\beta} \quad &: \quad \underset{\alpha,\beta}{sym}\left\{\rho^F X_\alpha'^F \frac{\partial \psi^F}{\partial L_{k\beta}^F}\right\} = 0,
\end{aligned}
\tag{3.7}
$$

(3.7cont)

and there remains a residual inequality which describes the dissipation in the system

$$
P_{k\alpha}^F L_{k\alpha}^F + \pi\left(v_K^F - v_K^S\right)\left(v_K^F - v_K^S\right) + \lambda n F_{\alpha k}^{S-1} L_{k\alpha}^F \geq 0. \tag{3.8}
$$

Consequently all but one multipliers are determined by the constitutive relations for the partial free energies. Solely the multiplier λ may be constitutively undetermined provided the coefficient ν consists of two contributions following from the relation $(3.7)_5$. In such a case we can extend the set of fields on λ which then plays the role of reaction force on the constraint.

On the other hand the reaction force on the constraint cannot contribute to the residual inequality because there is no dissipation due to holonomic constraints. It means that the partial Piola-Kirchhoff stress tensor $P_{k\alpha}^F$ must contain the contribution of the field λ of the following form

$$
P_{k\alpha}^F = -n\lambda F_{\alpha k}^{S-1} + \tilde{P}_{k\alpha}^F\left(n, F_{k\alpha}^S, L_{k\alpha}^F\right). \tag{3.9}
$$

Finally combining the remaining identities (3.7) we obtain the following constitutive relations for free energy functions

$$\psi^S = \psi^S\left(n, F^S_{k\alpha}\right), \qquad \psi^F = \psi^F\left(n\right). \tag{3.10}$$

The most important conclusion from the above considerations concerns the possibility of the existence of a thermodynamically admissible additional field λ similar to the reaction pressure in one-component models of incompressible materials. Its existence in the case considered in this note requires a contribution of the gradient of porosity to the source of momentum. Otherwise, for $\nu \equiv 0$, this quantity would be constitutively determined, and could not serve the purpose of an additional field.

4 Eulerian description

We summarize the results of the previous section in the eulerian description typical for models of porous, and granular materials as well as for suspentions. The balance equations have the following form

$$\begin{aligned}
\frac{\partial\left(1-n\right)}{\partial t} + \frac{\partial}{\partial x^k}\left[\left(1-n\right)v^S_k\right] &= 0, \\
\frac{\partial}{\partial x^k}\left[\left(1-n\right)v^S_k + n v^F_k\right] &= 0, \\
\left(1-n\right)\rho^{SR}\left(\frac{\partial v^S_k}{\partial t} + v^S_l\frac{\partial v^S_k}{\partial x^l}\right) &= \frac{\partial T^S_{kl}}{\partial x^l} + \pi\left(v^F_k - v^S_k\right) + \nu\frac{\partial n}{\partial x^k}, \\
n\rho^{FR}\left(\frac{\partial v^F_k}{\partial t} + v^F_l\frac{\partial v^F_k}{\partial x^l}\right) &= \frac{\partial T^F_{kl}}{\partial x^l} - \pi\left(v^F_k - v^S_k\right) - \nu\frac{\partial n}{\partial x^k}.
\end{aligned} \tag{4.1}$$

In these equations T^S_{kl} , and T^F_{kl} denote partial Cauchy stress tensors.

We skip here the discussion of the material objectivity which is standard. In addition we use the assumption of isotropy which has been already mentioned earlier in this work. Then for the quantities appearing in the above equations we have the following constitutive relations

$$\begin{aligned}
T^S_{kl} &= J^{S-1}P^S_{k\alpha}F^S_{l\alpha} = -\left(1-n\right)\lambda\delta_{kl} + \Im_0\delta_{kl} + \Im_1 B^S_{kl} + \Im_{-1}B^{S-1}_{kl}, \\
T^F_{kl} &= J^{S-1}P^F_{k\alpha}F^S_{l\alpha} = -n\lambda\delta_{kl} + \tilde{T}^F_{kl}\left(n, D^F_{kl}\right),
\end{aligned} \tag{4.2}$$

where the coefficients of the Cauchy stress tensor in the solid component are defined by the derivatives of the partial Helmholtz free energy ψ^S in the following way

$$
\begin{aligned}
\Im_0 \;&:\; = (1-n)\left[(1-n)\,\rho^{SR}\frac{\partial \psi^S}{\partial n} + n\rho^{FR}\frac{\partial \psi^F}{\partial n} + \right. \\
&\left. +2\rho^{SR}\left(\frac{\partial \psi^S}{\partial II}II + \frac{\partial \psi^S}{\partial III}III\right)\right], \qquad (4.3)\\
\Im_1 \;&:\; = 2\,(1-n)\,\rho^{SR}\frac{\partial \psi^S}{\partial I}, \qquad \Im_{-1} := -2\,(1-n)\,\rho^{SR}\frac{\partial \psi^S}{\partial II}III.
\end{aligned}
$$

The left Cauchy-Green deformation tensor B^S_{kl} , its invariants, and the symmetric part of the velocity gradient in the fluid component D^F_{kl} are defined as follows

$$
\begin{aligned}
B^S_{kl} \;&:\; = F^S_{k\alpha}F^S_{l\alpha}, \quad I := B^S_{kk}, \quad II := \frac{1}{2}\left(I^2 - B^S_{kl}B^S_{kl}\right),\\
III \;&:\; = \det B^S_{kl} = J^{S2}, \qquad (4.4)\\
D^F_{kl} \;&:\; = sym\left(L^F_{k\alpha}F^{S-1}_{\alpha l}\right).
\end{aligned}
$$

The contribution $\tilde{T}^F_{kl}\left(n, D^F_{kl}\right)$ to the stress tensor in the fluid is, of course, due to the viscosity of the fluid component on the macroscopic level.

Bearing the relations (3.9) in mind we have the following constitutive relations for the partial Helmholtz free energies, and for the coefficient in the contribution of porosity gradient to the source of momentum

$$
\begin{aligned}
\psi^S \;&=\; \psi^S\left(n, I, II, III\right), \quad \psi^F = \psi^F\left(n\right), \qquad (4.5)\\
\nu \;&=\; -\lambda + n\rho^{FR}\frac{\partial \psi^F}{\partial n}.
\end{aligned}
$$

Let us notice that in the case of free energy in the fluid component independent of porosity the coefficient ν reduces to $-\lambda$. If in addition the free energy ψ^S is independent of porosity as well then the model presented above is identical, for instance, with the model used in some theories of sedimentation (e.g. [4]). It is difficult to say if such an assumption is indeed physically justified. In such a case the multiplier λ plays the role of the pore pressure of classical models of soils.

On the other hand if the coefficient ν were identically zero the relation $(4.5)_3$ would reduce to the *constitutive law* for the multiplier λ, as we have already mentioned. Certainly in such a case the multiplier cannot be an additional field, and the system of field equations becomes overdetermined. This was the case for all models without contributions of higher gradients which I consider in the paper [1].

5 A simple example

In order to demonstrate flaws and discrepancies of models with and without higher gradients we consider a simple example. We compare results following from the model mentioned in the last part of the previous section with results of a model with compressible components . Let us investigate a static state of a tube of length L containing a granular material with a given varying porosity $n(x)$, where x is the variable along the tube. The tube is loaded by an equal pressure head p_0 at both ends. The momentum balance equations of the above discussed model with an inviscid fluid component yield

$$\begin{aligned} -\frac{\partial p^F}{\partial x} + \lambda \frac{\partial n}{\partial x} &\equiv -n\frac{\partial \lambda}{\partial x} = 0,\ T^F_{kl} = -p^F \delta_{kl} \implies \lambda = p_0,\ p^F = np_0, \\ \frac{\partial \sigma^S}{\partial x} - \lambda \frac{\partial n}{\partial x} &\equiv -(1-n)\frac{\partial \lambda}{\partial x} = 0 \qquad \implies \qquad \sigma^S = -(1-n)\, p_0, \end{aligned} \tag{5.1}$$

where σ^S is the component T^S_{11} of the Cauchy stress in the solid component. Hence the problem is statically determined. The mass density of the fluid component is determined by the real (constant) mass density: $\rho^F_t = n\rho^{FR}$, and it is independent of the loading. Similarly deformations of the solid follow alone from the constitutive law: $\rho^S_t = n\rho^{SR}$, and they are independent of the loading as well.

The problem is more involved in the case of a compressible model. The momentum balance equations and the simplest constitutive relations have in this case the following form (e.g. [7])

$$\begin{aligned} \frac{\partial p^F}{\partial x} &= 0, \quad \frac{\partial \sigma^S}{\partial x} = 0, \\ p^F &= c^{F2}(n)\,\rho^F_t, \quad \sigma^S = E^S(n)\,\epsilon^S, \end{aligned} \tag{5.2}$$

where c^F is the material parameter connected with the propagation of sound in the fluid component, and E^S is the elasticity coefficient of the solid component. ϵ^S denotes the elongation. The boundary conditions for the tube in the static case reduce to the following relations

$$\begin{aligned} p^F - \sigma^S\big|_{x=0} &= p^F - \sigma^S\big|_{x=L} = p_0, \\ p^F - np_0\big|_{x=0} &= p^F - np_0\big|_{x=L} = 0, \end{aligned} \tag{5.3}$$

where the last two relations follow from kinematic boundary conditions for the static case (e.g. see the formula (3.5) in [8]) This problem has a solution solely under the assumption

$$n\left(x=0\right)=n\left(x=L\right). \tag{5.4}$$

Otherwise a static solution does not exist. If the condition (5.4) is satisfied we obtain immediately

$$\begin{aligned} p^F &= n\left(x=0\right)p_0, \quad \rho_t^F=\frac{n\left(x=0\right)}{nc^{F2}\left(n\right)}p_0, \\ \sigma^S &= -\left(1-n\left(x=0\right)\right)p_0, \quad \epsilon^S=-\frac{\left(1-n\left(x=0\right)\right)}{E^S\left(n\right)}p_0. \end{aligned} \tag{5.5}$$

Hence both models deliver certain artefacts: the first one predicts constant mass densities of components in spite of the varying partial pressure and the partial stress, the second one requires the condition (5.4) for variations of porosity. However the results for the partial pressure in the fluid component, and for the partial stress in the solid component agree on the boundaries. It is difficult to predict how big would be discrepancies inside of the tube as such an experimental comparison was never performed.

6 Concluding remarks

Results presented in this note show that some models of multicomponent systems with the condition of incompressibility of real components can be made thermodynamically admissible. In order to achieve this result we had to introduce the gradient of porosity as a constitutive variable. Constitutive results remind these used in many papers on the subject of flows through porous or granular materials even though there are some modifications connected with a possible dependence of free energies on the porosity. However even in the simple version of the model presented in this work this assumption is connected with problems in the formulation of boundary conditions as dynamical compatibility conditions for momenta contain an explicit contribution of the jump of porosity. This requires an extension of the notion of porosity to an exterior which is usually connected with difficult considerations of boundary layers (see: [5]).

Apart from this typical weakness of a model with higher gradients we should not forget that practical applications of models of porous, and granular materials frequently require the compressibility of real components because some of them are either gases or mixtures of fluids with vapours. In such cases we need a field equation for the porosity anyway in order to close the system. This may be an evolution equation or some balance equation as proposed, for instance, by Goodman and Cowin [6], or by myself [7]. Such models do not require higher gradients as constitutive variables in order to be thermodynamically admissible.

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